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Optimal control of the fuel reload mechanism ⋆

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Abstract: The algorithm for determining the optimal route for moving and controlling the mechanisms for reloading fuel assemblies of fast neutron reactors is proposed. It allows increasing the efficiency of the Nuclear Power Plant operation by reducing the stopping time for nuclear fuel transhipment.

Keywords: Optimal control, fuel reload mechanism.

1. INTRODUCTION

The necessity of periodic reloading of fuel assemblies with nuclear fuel (once in 1.0-1.5 years) is a feature of Nuclear Power Plant (NPP) operation. The overload of nuclear fuel includes the unloading of spent fuel assemblies, their rearrangement in the core, the loading of fresh fuel assemblies (see Beltyukov (2013)). Reduction in the duration of the shutdown of the power unit for the transfer of nuclear fuel increases the amount of electricity generated and, accordingly, the efficiency of the use of nuclear power plants. Minimizing the time of transfer of the capture from the initial position to the specified point will help reduce the time spent on fuel overload. This, in turn, will reduce the reactor downtime due to the overload of nuclear fuel. Overloading of reactors on fast neutrons with a sodium coolant should occur without contact of sodium with air. Therefore, the capture of the grip on the coordinates of the fuel assembly to be reloaded is carried out by means of two eccentrically located rotating plugs - Fig. 1. Two rotary plugs (the smaller of them located inside a large plug) are located on the neck of the reactor vessel. The fuel reload mechanism moving in a vertical direction (extraction and installation in a cell). It located on a small plug. The plugs, which are the reactor cover, act as thermal and biological protection, and also place the overload mechanism on the given coordinates of the core in order to capture the fuel assembly and move it to the required area with the given coordinates. Note that related problems arise in robotics (see Chernoushko (1989), Avetisyan (1985)). This paper is devoted to the task of minimizing the time for the capture, located on a smaller plug, to a given fuel assembly. Solving this problem will help reduce the stoppage time of the power unit for performing fuel transshipment operations.

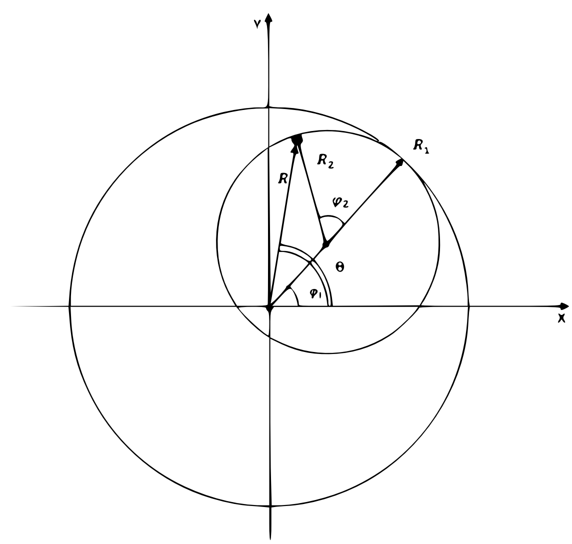

Fig. 1. Scheme of rotary plugs

2. A MATHEMATICAL MODEL DESCRIBING THE DYNAMICS OF PLUGS

The mechanical system consists of two rotary plugs (Fig. 1). A large plug is a disk of radius R_1, whose geometric center remains stationary during the motion. The large plug has an eccentric circular cutout of radius R_2, inside

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of which is placed a small plug - a disk of radius R_2, whose center of mass lies at a distance e_2 from its geometric center (see Fig. 1) When describing the interaction of plugs, we neglect frictional forces. The motion of the mechanical system is flat. The axial moment of inertia of a large plug is J_1, and the small plug relative to its center of mass is J_2, the mass of the second plug is m_2. The distance between the geometric centers of the large and small congestions is e_1. To describe the dynamics of the mechanical system with two degrees of freedom, we use the Lagrange equations of the second kind Appell (1893). As generalized coordinates, we select the angle of rotation of the large plugs — φ_1 and the angle of rotation of the small plugs — φ_2 with respect to the large plugs. The generalized forces are the control moments u_1 and u_2, applied to the large and small plugs, respectively. The kinetic energy of the system is given by

$$T = \frac{1}{2}\left((J_1 + J_2 + m_2e_1^2)\dot\varphi_1^2 + (J_2 + m_2e_2^2)\dot\varphi_2^2 \right. \\ \left. +2(J_2 + m_2e_1e_2\cos(\varphi_2))\dot\varphi_1\dot\varphi_2\right).$$

The mathematical model of a reloading device is described with the help of the following Lagrange equations of the second kind:

$$\left(J_1 + J_2 + m_2e_1^2\right)\ddot\varphi_1 + (J_2 + m_2e_1e_2\cos(\varphi_2))\ddot\varphi_2 \\ -m_2e_1e_2\sin(\varphi_2)\dot\varphi_2^2 = u_1, \\ (J_2 + m_2e_1e_2\cos(\varphi_2))\ddot\varphi_1 + \left(J_2 + m_2e_2^2\right)\ddot\varphi_2 = u_2. \quad (1)$$

The control moments applied to the large and small plugs, respectively, satisfy the constraints

$$|u_1| \le \mu_1, |u_2| \le \mu_2. \quad (2)$$

The mathematical model of a reloading device can be described using canonical equations, using generalized impulses

$$p_1 = \frac{\partial T}{\partial\dot\varphi_1} = (J_1+J_2+m_2e_1^2)\dot\varphi_1+(J_2+m_2e_1e_2\cos(\varphi_2))\dot\varphi_2,$$

$$p_2 = \frac{\partial T}{\partial\dot\varphi_2} = (J_2 + m_2e_1e_2\cos(\varphi_2))\dot\varphi_1 + (J_2 + m_2e_2^2)\dot\varphi_2.$$

The Hamiltonian function is given by

$$H = \frac{1}{2\Delta(\varphi_2)}\left((J_2 + m_2e_2^2)p_1^2 \right. \\ \left. -2(J_2 + m_2e_1e_2\cos(\varphi_2))p_1p_2 + (J_1 + J_2 + m_2e_1^2)p_2^2\right),$$

where

$$\Delta(\varphi_2) = (J_1+J_2+m_2e_1^2)(J_2+m_2e_2^2)-(J_2+m_2e_1e_2\cos(\varphi_2))^2.$$

The canonical equations have the form

$$\dot\varphi_1 = \frac{1}{\Delta(\varphi_2)}\left((J_2 + m_2e_2^2)p_1 - (J_2 + m_2e_1e_2\cos(\varphi_2))p_2\right),$$

$$\dot\varphi_2 = \frac{1}{\Delta(\varphi_2)}\left(-(J_2 + m_2e_1e_2\cos(\varphi_2))p_1 \right. \\ \left. +(J_1 + J_2 + m_2e_1^2)p_1\right), \\ \dot p_1 = u_1, \quad (3)$$

$$\dot p_2 = u_2 + \frac{m_2e_1e_2\sin(\varphi_2)}{\Delta^2(\varphi_2)}(J_2 + m_2e_1e_2\cos(\varphi_2)) \\ \times\left((J_2 + m_2e_2^2)p_1^2 + (J_1 + J_2 + m_2e_1^2)p_2^2\right) \\ -\frac{m_2e_1e_2\sin(\varphi_2)}{\Delta^2(\varphi_2)}(J_2 + m_2e_2^2)\left((J_1 + J_2 + m_2e_1^2) \right. \\ \left. \times(J_2 + m_2e_2^2) + (J_2 + m_2e_1e_2\cos(\varphi_2))^2\right)p_1p_2.$$

When the center of mass of the second plug coincides with its geometric center e_2, the system of canonical equations has the form

$$\frac{d\varphi_1}{dt} = \alpha_1\left(p_1 - p_2\right), \ \frac{d\varphi_2}{dt} = -\alpha_1\left(p_1 - p_2\right) + \alpha_2p_2, \\ \frac{dp_1}{dt} = u_1, \ \frac{dp_2}{dt} = u_2, \quad (4)$$

where $\alpha_1 = \left(J_1 + m_2e_1^2\right)^{-1}$, $\alpha_2 = J_2^{-1}$.

We consider the problem of moving the phase point of the system (4) from the initial position $(0, 0, 0, 0)^\top$ to the final position $(\varphi_1^*, \varphi_2^*, 0, 0)^\top$, $\varphi_1^*, \varphi_2^* \in \mathbf{R}$.

3. THE SEQUENTIAL PROCEDURE FOR CONTROLLING THE FUEL RELOAD MECHANISM

In this section, we will consider the procedure for sequential control of the fuel reload mechanism. First, one stopper will be rotated, followed by a second stopper.

First we consider the auxiliary optimal speed problem for the system

$$\frac{d\varphi}{dt} = \alpha p, \ \frac{dp}{dt} = u. \quad (5)$$

It is required to move the phase point of the system (5) from position $\left(\varphi^1, 0\right)^\top$ to position $\left(\varphi^2, 0\right)^\top$, $\varphi^1 \ne \varphi^2$, $\varphi^1, \varphi^2 \in \mathbf{R}$ by means of a control u that satisfies the constraint $|u| \le \mu$, $\mu > 0$ or the shortest time. Here α is a positive constant.

Theorem 1. The optimal control for the above formulated problem has a unique switching and is determined by the formula

$$u(t) = -\mu\mathrm{sgn}(\varphi^2 - \varphi^1)\mathrm{sgn}(t - \vartheta/2), \ t \in [0, \vartheta], \quad (6)$$

where

$$\vartheta = 2\sqrt{\frac{|\varphi^2 - \varphi^1|}{\mu\alpha}} \quad (7)$$

is the optimal time for the time optimal problem for the system (5).

Proof. The system (5) is completely controllable and general conditions are fulfilled for it. Therefore, the problem has a unique solution. To find it, we use the Pontryagin maximum principle Pontryagin (1964); Li (1967). We introduce the function

$$H(\varphi, p, \psi_1, \psi_2) = \alpha p\psi_1 + u\psi_2$$

write the conjugate system

$$\frac{d\psi_1}{dt} = 0, \ \frac{d\psi_2}{dt} = \alpha\psi_1. \quad (8)$$

The solutions of the system (8) are defined by formulas

$$\psi_1(t) = \psi_1^0, \ \psi_2(t) = \psi_2^0 + \alpha\psi_1^0 t, \ t \in [0, \vartheta]. \quad (9)$$

Using the Pontryagin maximum principle Pontryagin (1964), we obtain the following formula for optimal control of the system (5)

$$u(t) = \mu\mathrm{sgn}(\psi_2(t)), \ t \in [0, \vartheta]. \quad (10)$$

From the system (5) we find

$$p(t) = \mu \int_0^t \operatorname{sgn}(\psi_2(s))ds,\ t \in [0, \vartheta]. \tag{11}$$

Taking into account the conditions $p(0) = p(\vartheta) = 0$, we conclude from (11) that $\psi_1^0 \neq 0, \psi_2^0 \neq 0$ and there exist a single switching point for the control $0 < \vartheta_* < \vartheta$. From (9) and (11) we find $\psi_2(t) = \alpha\psi_1^0(t - \vartheta_*)$, $t \in [0, \vartheta]$, and

$$p(t) = -\mu\alpha \operatorname{sgn}(\psi_1^0)t,\ t \in [0, \vartheta_*),$$
$$p(t) = -\mu\alpha \operatorname{sgn}(\psi_1^0)(t - 2\vartheta_*),\ t \in [\vartheta_*, \vartheta].$$

Taking into account the condition $p(\vartheta) = 0$, we find $\vartheta_* = \vartheta/2$. As a result, we have

$$p(t) = -\mu\alpha \operatorname{sgn}(\psi_1^0)t,\ t \in [0, \vartheta/2),$$
$$p(t) = -\mu\alpha \operatorname{sgn}(\psi_1^0)(t - \vartheta),\ t \in [\vartheta/2, \vartheta]. \tag{12}$$

From the system (5) we find

$$\varphi(t) = \varphi^1 - \mu\alpha \operatorname{sgn}(\psi_1^0)t^2/2,\ t \in [0, \vartheta/2),$$
$$\varphi(t) = \varphi^1 - \mu\alpha \operatorname{sgn}(\psi_1^0)$$
$$\times(\vartheta^2/4 - (t - \vartheta)^2/2),\ t \in [\vartheta/2, \vartheta]. \tag{13}$$

Taking into account the condition $\varphi(\vartheta) = \varphi^2 = \varphi^1 - \mu\alpha \operatorname{sgn}(\psi_1^0)\vartheta^2/4$, we find $\operatorname{sgn}(\psi_1^0) = -\operatorname{sgn}(\varphi^2 - \varphi^1)$ and $\vartheta = 2\sqrt{|\varphi^2 - \varphi^1|/(\mu\alpha)}$. As a result, the optimal control for the system (5) is determined by the formula (6), and its optimal solutions by formulas

$$\varphi(t) = \varphi^1 + \mu\alpha \operatorname{sgn}(\varphi^2 - \varphi^1)t^2/2,\ t \in [0, \vartheta/2),$$
$$\varphi(t) = \varphi^1 + \mu\alpha \operatorname{sgn}(\varphi^2 - \varphi^1)$$
$$\times(\vartheta^2/4 - (t - \vartheta)^2/2),\ t \in [\vartheta/2, \vartheta], \tag{14}$$

$$p(t) = \mu \operatorname{sgn}(\varphi^2 - \varphi^1)t,\ t \in [0, \vartheta/2),$$
$$p(t) = \mu \operatorname{sgn}(\varphi^2 - \varphi^1)(\vartheta - t),\ t \in [\vartheta/2, \vartheta]. \tag{15}$$

Consider a sequential procedure for controlling the fuel reload mechanism. At the first stage of control on the time interval $[0, \vartheta_1]$ only the first control u_1 acts, and in the second stage of control ($t \in [\vartheta_1, \vartheta_1 + \vartheta_2]$) only the second control u_2 in not equal to zero.

Theorem 2. The controls implementing a sequential procedure for moving the system (4) from the initial position $(0, 0, 0, 0)^\top$ to the final position $(\varphi_1^*, \varphi_2^*, 0, 0)^\top$, have two switching times and are defined by formulas

$$u_1(t) = -\mu_1 \operatorname{sgn}(\varphi_1^* + \frac{\alpha_1}{\alpha_2}(\varphi_1^* + \varphi_2^*))\operatorname{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1),$$
$$u_1(t) = 0,\ t \in [\vartheta_1, \vartheta_1 + \vartheta_2],$$
$$u_2(t) = 0,\ t \in [0, \vartheta_1),$$
$$u_2(t) = -\mu_2 \operatorname{sgn}(\varphi_1^* + \varphi_2^*)\operatorname{sgn}(t - \vartheta_1 - \vartheta_2/2),\ t \in [\vartheta_1, \vartheta_1 + \vartheta_2],$$

where

$$\vartheta_1 = \frac{2}{\sqrt{\mu_1}}\sqrt{|(\alpha_1^{-1} + \alpha_2^{-1})\varphi_1^* + \alpha_2^{-1}\varphi_2^*|},$$
$$\vartheta_2 = \frac{2}{\sqrt{\mu_2}}\sqrt{|\alpha_2^{-1}(\varphi_1^* + \varphi_2^*)|}.$$

Proof. In the first stage of the control, we have $u_2(t) \equiv 0$, $t \in [0, \vartheta_1]$. Then the initial problem reduces to a simple problem of optimal performance for the auxiliary system

$$\frac{d\varphi_1}{dt} = \alpha_1 p_1,\ \frac{dp_1}{dt} = u_1. \tag{16}$$

The phase point must be move from the starting position $(0, 0)^\top$ to the final position $(\widetilde{\varphi}_1^*, 0)^\top$, using controls that satisfy the geometric constraint $|u_1| \leq \mu_1$. Here $\widetilde{\varphi}_1^*$ is the unknown coordinate of the final position.

Using the theorem 1, we find the optimal control for the auxiliary system (16)

$$u_1(t) = -\mu_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)\operatorname{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1], \tag{17}$$

where

$$\vartheta_1 = 2\sqrt{|\widetilde{\varphi}_1^*|/(\mu_1\alpha_1)} \tag{18}$$

is an optimum time of the optimum speed problem for the system (16). Using the theorem 1, we find the optimal solution for the auxiliary system (16)

$$\varphi_1(t) = \mu_1\alpha_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)t^2/2,\ t \in [0, \vartheta_1/2),$$
$$\varphi_1(t) = \mu_1\alpha_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)(\vartheta_1^2/4 - (t - \vartheta_1)^2/2),\ t \in [\vartheta_1/2, \vartheta_1],$$
$$p_1(t) = \mu_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)t,\ t \in [0, \vartheta_1/2),$$
$$p1(t) = \mu_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)(\vartheta_1 - t),\ t \in [\vartheta_1/2, \vartheta_1]. \tag{19}$$

The law of variation of φ_2 is found from the boundary value problem

$$\frac{d\varphi_2}{dt} = -\alpha_1 p_1,\ \varphi_2(0) = 0.$$

As a result, we have

$$\varphi_2(t) = -\mu_1\alpha_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)t^2/2,\ t \in [0, \vartheta_1/2),$$
$$\varphi_2(t) = -\mu_1\alpha_1 \operatorname{sgn}(\widetilde{\varphi}_1^*)(\vartheta_1^2/4$$
$$-(t - \vartheta_1)^2/2),\ t \in [\vartheta_1/2, \vartheta_1]. \tag{20}$$

In the second stage of control, we have $u_1(t) \equiv 0$, $t \in [\vartheta_1, \vartheta_1 + \vartheta_2$. Using the time change $t + \vartheta_1 \to t$, we reduce the original problem to a simple time optimal problem for the auxiliary system

$$\frac{d\varphi_2}{dt} = (\alpha_1 + \alpha_2)p_2,\ \frac{dp_2}{dt} = u_2. \tag{21}$$

The phase point of the system must be moving from the position $(\widetilde{\varphi}_2, 0)^\top$ to the position$(\varphi_2^*, 0)^\top$, using controls that satisfy the geometric constraint $|u_2| \leq \mu_2$. Here it is assumed that $\widetilde{\varphi}_2 = \varphi_2(\vartheta_1)$.

Using the theorem 1, we find the optimal control for the auxiliary system (21)

$$u_1(t) = -\mu_2 \operatorname{sgn}(\varphi_2^* - \widetilde{\varphi}_2)\operatorname{sgn}(t - \vartheta_2/2),\ t \in [0, \vartheta_2], \tag{22}$$

where

$$\vartheta_2 = 2\sqrt{|\varphi_2^* - \widetilde{\varphi}_2|/(\mu_2(\alpha_1 + \alpha_2))} \tag{23}$$

is an optimal time of the optimum speed problem for the system (21). Using the theorem 1, we find the optimal solution for the auxiliary system (21)

$$\varphi_2(t) = \widetilde{\varphi}_2 + \mu_2(\alpha_1 + \alpha_2)\operatorname{sgn}(\varphi_2^* - \widetilde{\varphi}_2)t^2/2,\ t \in [0, \vartheta_2/2),$$
$$\varphi_2(t) = \widetilde{\varphi}_2 + \mu_2(\alpha_1 + \alpha_2)\operatorname{sgn}(\varphi_2^* - \widetilde{\varphi}_2)(\vartheta_2^2/4$$

$$-(t-\vartheta_2)^2/2),\ t\in[\vartheta_2/2,\vartheta_2], \qquad (24)$$

$$p_2(t)=\mu_2\mathrm{sgn}(\varphi_2^*-\widetilde{\varphi}_2)t,\ t\in[0,\vartheta_2/2),$$
$$p_2(t)=\mu_2\mathrm{sgn}(\varphi_2^*-\widetilde{\varphi}_2)(\vartheta_2-t),\ t\in[\vartheta_2/2,\vartheta_2]. \qquad (25)$$

The law of variation of φ_1 is found from the boundary value problem

$$\frac{d\varphi_1}{dt}=-\alpha_1 p_2,\ \varphi_1(\vartheta_2)=\varphi_1^*.$$

We have

$$\varphi_1(t)=\varphi_1^*+\mu_2\alpha_1\mathrm{sgn}(\varphi_2^*-\widetilde{\varphi}_2)(\vartheta_2^2/4-t^2/2),\ t\in[0,\vartheta_1/2),$$
$$\varphi_1(t)=\varphi_1^*+\mu_2\alpha_1\mathrm{sgn}(\varphi_2^*-\widetilde{\varphi}_2)$$
$$\times(t-\vartheta_2)^2/2,\ t\in[\vartheta_2/2,\vartheta_2]. \qquad (26)$$

From (19) and (26) we find the continuity conditions for the angular variable φ_1

$$\mu_1\alpha_1\mathrm{sgn}(\widetilde{\varphi}_1^*)\vartheta_1^2/4=\varphi_1^*+\mu_2\alpha_1\mathrm{sgn}(\varphi_2^*-\widetilde{\varphi}_2)\vartheta_2^2/4. \qquad (27)$$

From (20) and (24) we find the continuity conditions for the angular variable φ_2

$$-\mu_1\alpha_1\mathrm{sgn}(\widetilde{\varphi}_1^*)\vartheta_1^2/4=\widetilde{\varphi}_2. \qquad (28)$$

Using the formulas (18) and (23), we get $\widetilde{\varphi}_2=-\widetilde{\varphi}_1^*$, $\widetilde{\varphi}_1^*=\varphi_1^*+(\alpha_1/\alpha_2)(\varphi_1^*+\varphi_2^*)$. As a result, we find the representations of the controls u_1, u_2 and the values ϑ_1, ϑ_2, indicated in the statement of the theorem 2.

From theorem 2 it follows that for a sequential control procedure for the fuel reload mechanism, the time of moving from the initial position $(0,0,0,0)^\top$ to the final position $(\varphi_1^*,\varphi_2^*,0,0)^\top$ is given by

$$\vartheta_{poc}=\frac{2}{\sqrt{\mu_1}}\sqrt{|(\alpha_1^{-1}+\alpha_2^{-1})\varphi_1^*+\alpha_2^{-1}\varphi_2^*|}$$
$$+\frac{2}{\sqrt{\mu_2}}\sqrt{|\alpha_2^{-1}(\varphi_1^*+\varphi_2^*)|}.$$

4. PARALLEL PROCEDURE FOR CONTROLLING THE FUEL RELOAD MECHANISM

For the original system (4), the function $\varphi=\varphi_1+\varphi_2$ can be determined from the time optimal problem for the system

$$\frac{d\varphi}{dt}=\alpha_2 p_2,\ \frac{dp_2}{dt}=u_2, \qquad (29)$$

which must be moving the starting position $(0,0)^\top$ to the final position $(\varphi^*,0)^\top$, where $\varphi^*=\varphi_1^*+\varphi_2^*$, using controls satisfying a geometric constraint $|u_2|\le\mu_2$.

Applying the theorem, we find that the optimal time for the time optimal problem for the system (29) is given by

$$\widetilde{\vartheta_2}=\frac{2}{\sqrt{\mu_2}}\sqrt{\frac{|\varphi_1^*+\varphi_2^*|}{\alpha_2}}, \qquad (30)$$

those. coincides with the time ϑ_2 of the auxiliary optimal speed problem for the system (21) of the sequential control procedure. The optimal control for the system (29) is given by formula

$$u_2(t)=-\mu_2\mathrm{sgn}(\varphi_1^*+\varphi_2^*)\mathrm{sgn}(t-\vartheta_2/2),\ t\in[0,\vartheta_2], \qquad (31)$$

Using the theorem 1, we find the optimal solution for the system (29)

$$\varphi(t)=\mu_2\alpha_2\mathrm{sgn}(\varphi_1^*+\varphi_2^*)t^2/2,\ t\in[0,\vartheta_2/2),$$
$$\varphi(t)=\mu_2\alpha_2\mathrm{sgn}(\varphi_1^*+\varphi_2^*)(\vartheta_2^2/4$$
$$-(t-\vartheta_2)^2/2),\ t\in[\vartheta_2/2,\vartheta_2], \qquad (32)$$

$$p_2(t)=\mu_2\mathrm{sgn}(\varphi_1^*+\varphi_2^*)t,\ t\in[0,\vartheta_2/2),$$
$$p_2(t)=\mu_2\mathrm{sgn}(\varphi_1^*+\varphi_2^*)(\vartheta_2-t),\ t\in[\vartheta_2/2,\vartheta_2]. \qquad (33)$$

To find the control u_1, we use the terminal problem on the interval $[0,\vartheta_2]$ for the system

$$\frac{d\varphi_1}{dt}=\alpha_1(p_1-p_2(t)),\ \frac{dp_1}{dt}=u_1, \qquad (34)$$

which must be moving from the starting position $(0,0)^\top$ to the final position $(\varphi_1^*,0)^\top$, using controls satisfying the geometric constraint $|u_1|\le\mu_1$.

Integrating the system (34), we find

$$p_1(t)=\int_0^t u_1(s)ds,\ t\in[0,\vartheta_2], \qquad (35)$$

$$\varphi_1(t)=\alpha_1\int_0^t(t-s)u_1(s)ds$$
$$-\alpha_1\int_0^t p_2(s)ds,\ t\in[0,\vartheta_2]. \qquad (36)$$

The solutions obtained must satisfy the following conditions:

$$p_1(\vartheta_2)=\int_0^{\vartheta_2}u_1(s)ds=0,$$

$$\varphi_1(\vartheta_2)=\varphi_1^*=\alpha_1\int_0^{\vartheta_2}(\vartheta_2-s)u_1(s)ds-\alpha_1\int_0^{\vartheta_2}p_2(s)ds.$$

As a result, the admissible controls u_1 of the terminal task in question must satisfy the following requirements:

$$|u_1(t)|\le\mu_1,\ t\in[0,\vartheta_2],\ \int_0^{\vartheta_2}u_1(s)ds=0,$$

$$\int_0^{\vartheta_2}(s-\vartheta_2/2)u_1(s)ds=-\Delta_2, \qquad (37)$$

where $\Delta_2=\varphi_1^*/\alpha_1+\int_0^{\vartheta_2}p_2(s)ds=\varphi_1^*/\alpha_1+\mathrm{sgn}(\varphi_1^*+\varphi_2^*)\vartheta_2^2/4=\varphi_1^*/\alpha_1+(\varphi_1^*+\varphi_2^*)/\alpha_2$.

We shall find the conditions under which the control

$$u_1(t) = \beta_1 \mathrm{sgn}(t - \vartheta_2/2),\ t \in [0, \vartheta_2], \tag{38}$$

satisfies the requirements (37). We have $|\beta_1| \le \mu_1$, $\beta_1 = -4\Delta_2/\vartheta_2^2$. Consequently, the requirements (37) are satisfied by a single relay control of the form (39)

$$u_1(t) = -\frac{\mu_2\alpha_2}{|\varphi_1^* + \varphi_2^*|}(\varphi_1^*/\alpha_1 + (\varphi_1^* + \varphi_2^*)/\alpha_2)\,\mathrm{sgn}(t - \vartheta_2/2),\ t \in [0, \vartheta_2], \tag{39}$$

if condition

$$\left|\frac{\varphi_1^*}{\alpha_1} + \frac{\varphi_1^* + \varphi_2^*}{\alpha_2}\right| \le \frac{\mu_1}{\mu_2}\frac{|\varphi_1^* + \varphi_2^*|}{\alpha_2}. \tag{40}$$

is holds. The condition (40) imposes restrictions on the set Φ_1 of final position of the system.

We show that for any control u_1 satisfying the requirements (37), it is satisfied. Indeed, for such control the inequality

$$|\Delta_2| = \left|\int_0^{\vartheta_2}(s - \vartheta_2/2)u_1(s)ds\right| \le \mu_1 \int_0^{\vartheta_2} |\vartheta_2/2 - s| ds = \mu_1\vartheta_2^2/2, \tag{41}$$

which is equivalent to the inequality (40). Consequently, for finite positions not belonging to the set Φ_1, there are no controls for solving the terminal problem for the system (34)

Consider another substitution. We can determine the function $\chi = \varphi_1 + \frac{\alpha_1}{\alpha_1+\alpha_2}\varphi_2$. Now we will consider the time optimal problem for the system

$$\frac{d\chi}{dt} = \frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}p_1,\ \frac{dp_1}{dt} = u_1. \tag{42}$$

The phase point of system (42) must be transferred from the initial position $(0,0)^\top$ to the final position $(\chi^*, 0)^\top$, where $\chi^* = \varphi_1^* + \frac{\alpha_1}{\alpha_1+\alpha_2}\varphi_2^*$, using controls that satisfy the geometric constraint $|u_1| \le \mu_1$.

Applying the theorem 1 to this problem, we find that the optimal time in the time optimal problem for the system (42) is given by

$$\widetilde{\vartheta_1} = \frac{2}{\sqrt{\mu_1}}\sqrt{\frac{(\alpha_1+\alpha_2)|\chi^*|}{\alpha_1\alpha_2}} = \frac{2}{\sqrt{\mu_1}}\sqrt{|(\alpha_1^{-1} + \alpha_2^{-1})\varphi_1^* + \alpha_2^{-1}\varphi_2^*|}. \tag{43}$$

This time coincides with the time ϑ_1 of the auxiliary optimal speed problem for system (16) of the sequential control procedure. The optimal control for system (42) is determined by formula

$$u_1(t) = -\mu_1 \mathrm{sgn}(\chi^*)\mathrm{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1]. \tag{44}$$

Using the theorem 1, we find the optimal solution for the system (42)

$$\chi(t) = \frac{\mu_1\alpha_1\alpha_2}{\alpha_1+\alpha_2}\mathrm{sgn}(\chi^*)t^2/2,\ t \in [0, \vartheta_1/2),$$

$$\chi(t) = \frac{\mu_1\alpha_1\alpha_2}{\alpha_1+\alpha_2}\mathrm{sgn}(\chi^*)(\vartheta_1^2/4 - (t-\vartheta_1)^2/2),\ t \in [\vartheta_1/2, \vartheta_1], \tag{45}$$

$$p_1(t) = \mu_1 sgn(\chi^*)t,\ t \in [0, \vartheta_1/2),\ p_1(t) = \mu_1 \mathrm{sgn}(\chi^*)(\vartheta_1 - t),\ t \in [\vartheta_1/2, \vartheta_1]. \tag{46}$$

To find the optimal control u_2 , we use the terminal problem on a segment $[0, \vartheta_1]$ for the system

$$\frac{d\varphi_2}{dt} = (\alpha_1 + \alpha_2)p_2 - \alpha_1 p_1(t),\ \frac{dp_2}{dt} = u_2. \tag{47}$$

This system must be transferred from the initial position $(0,0)^\top$ to the final position $(\varphi_2^*, 0)^\top$ using a control that satisfies the geometric constraint $|u_2| \le \mu_2$.

Integrating the system (47), we find

$$p_2(t) = \int_0^t u_2(s)ds,\ t \in [0, \vartheta_1], \tag{48}$$

$$\varphi_2(t) = (\alpha_1 + \alpha_2)\int_0^t (t - s)u_2(s)ds - \alpha_1 \int_0^t p_1(s)ds,\ t \in [0, \vartheta_1]. \tag{49}$$

The solutions obtained must satisfy the conditions

$$p_1(\vartheta_1) = \int_0^{\vartheta_1} u_2(s)ds = 0,$$

$$\varphi_2(\vartheta_1) = \varphi_2^* = (\alpha_1+\alpha_2)\int_0^{\vartheta_1}(\vartheta_1 - s)u_1(s)ds - \alpha_1\int_0^{\vartheta_1} p_1(s)ds.$$

As a result, the admissible controls u_2 of the terminal problem under consideration must satisfy the conditions

$$|u_2(t)| \le \mu_2,\ t \in [0, \vartheta_1],\ \int_0^{\vartheta_1} u_1(s)ds = 0,$$

$$\int_0^{\vartheta_1}(s - \vartheta_1/2)u_2(s)ds = -\Delta_1, \tag{50}$$

where $\Delta_1 = (\varphi_2^* + \alpha_1 \int_0^{\vartheta_1} p_1(s)ds)/(\alpha_1 + \alpha_2) = (\varphi_2^* + \alpha_1\mu_1 \mathrm{sgn}(\chi^*)\vartheta_1^2/4)/(\alpha_1 + \alpha_2) = (\varphi_1^* + \varphi_2^*)/\alpha_2$.

Now we find the conditions under which the control

$$u_2(t) = \beta_2 \mathrm{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1], \tag{51}$$

satisfies conditions (50). We have $|\beta_2| \le \mu_2, \beta_2 = -4\Delta_1/\vartheta_1^2$. Consequently, conditions (50) are satisfied by a single relay control of the form (51)

$$u_2(t) = -\frac{\mu_1(\varphi_1^* + \varphi_2^*)}{|\varphi_1^*(1 + \alpha_2/\alpha_1) + \varphi_2^*|}$$

$$\times \mathrm{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1], \tag{52}$$

if condition

$$\left|\frac{\varphi_1^*}{\alpha_1} + \frac{\varphi_1^* + \varphi_2^*}{\alpha_2}\right| \geq \frac{\mu_1}{\mu_2}\frac{|\varphi_1^* + \varphi_2^*|}{\alpha_2}. \tag{53}$$

is hold. Inequality (53) imposes restrictions on the set Φ_2 of final position of the system.

We show that for any control u_2 that satisfies (50), it is satisfied. Indeed, for such a control we have an inequality

$$|\Delta_1| = \left|\int_0^{\vartheta_1} (s - \vartheta_1/2) u_2(s) ds\right|$$

$$\leq \mu_2 \int_0^{\vartheta_{12}} |\vartheta_1/2 - s| ds = \mu_1 \vartheta_1^2/2. \tag{54}$$

This inequality is equivalent to inequality (53). Consequently, for finite positions that do not belong to the set Φ_2, there are no controls for the terminal problem for the system (47).

An arbitrary finite position belongs to the domain (40) or (53). For their intersection points, the equality $\vartheta_1 = \vartheta_2$ is valid. As a result, we have the theorem.

Theorem 3. There exist controls with one switching point that translate the system (4) from the initial position $(0, 0, 0, 0)^\top$ to the end position $(\varphi_1^*, \varphi_2^*, 0, 0)^\top$ for the time $\vartheta_{paral} = \vartheta_1$, $(\varphi_1^*, \varphi_2^*)^\top \in \Phi_1$, $\vartheta_{paral} = \vartheta_2$, $(\varphi_1^*, \varphi_2^*)^\top \in \Phi_2$, which is less than $\vartheta_{consistent}$ for a sequential control procedure for the fuel assembly overload mechanism (here $\vartheta_{paral} = \vartheta_1$ and $\vartheta_{paral} = \vartheta_2$ are the optimal time for a parallel displacement scheme for the sets of final positions Φ_1 and Φ_2, $\vartheta_{consistent}$ is the optimal time for a sequential displacement scheme). For $(\varphi_1^*, \varphi_2^*)^\top \in \Phi_1$ these controls are defined by the formulas:

$$u_1(t) = -\frac{\mu_2 \alpha_2}{|\varphi_1^* + \varphi_2^*|}$$

$$\times (\varphi_1^*/\alpha_1 + (\varphi_1^* + \varphi_2^*)/\alpha_2)\, \mathrm{sgn}(t - \vartheta_2/2),\ t \in [0, \vartheta_2],$$

$$u_2(t) = -\mu_2 \mathrm{sgn}(\varphi_1^* + \varphi_2^*) \mathrm{sgn}(t - \vartheta_2/2),\ t \in [0, \vartheta_2],$$

and for $(\varphi_1^*, \varphi_2^*)^\top \in \Phi_2$ with the help of formulas:

$$u_1(t) = -\mu_1 \mathrm{sgn}((\alpha_1 + \alpha_2)\varphi_1^* + \alpha_1 \varphi_2^*)$$

$$\times \mathrm{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1],$$

$$u_2(t) = -\frac{\mu_1(\varphi_1^* + \varphi_2^*)}{|\varphi_1^*(1 + \alpha_2/\alpha_1) + \varphi_2^*|}$$

$$\times \mathrm{sgn}(t - \vartheta_1/2),\ t \in [0, \vartheta_1].$$

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